Vector Spaces – Core Concepts

# 📦 Vector Spaces

A vector space over a field (e.g., ℝ) is a set of vectors closed under vector addition and scalar multiplication that satisfies 8 axioms like associativity, distributivity, additive identity, etc.

# 🔗 Linear Independence

A set of vectors {v₁, ..., vₙ} is linearly independent if no vector can be written as a linear combination of the others. If some vector is dependent on others, they are linearly dependent.

# 🧱 Basis and Rank

• A basis is a linearly independent set that spans the vector space.  
• The number of vectors in a basis = dimension of the space.  
• Rank of a matrix = number of linearly independent rows/columns = dimension of the column space.

# 🧭 Affine Spaces

An affine space is like a vector space but without a fixed origin. It allows translation and is defined as a set of points formed by adding a fixed point to all vectors in a subspace.

# 📏 Norms

A norm assigns a non-negative length or size to each vector. Common norms:  
• L2 norm (Euclidean): ‖x‖₂ = √(x₁² + x₂² + ... + xₙ²)  
• L1 norm (Manhattan): ‖x‖₁ = |x₁| + |x₂| + ... + |xₙ|  
• L∞ norm (Max): ‖x‖∞ = max(|xᵢ|)

# 🎯 Inner Products

The inner product (dot product in ℝⁿ) measures similarity:  
⟨x, y⟩ = x₁y₁ + x₂y₂ + ... + xₙyₙ  
It is used to define length, angle, and orthogonality.

# 📐 Lengths and Distances

• Length of a vector = ‖x‖ = √(⟨x, x⟩)  
• Distance between two vectors x and y = ‖x − y‖

# 📐 Angles and Orthogonality

The angle θ between two vectors x and y is given by:  
cos(θ) = ⟨x, y⟩ / (‖x‖ \* ‖y‖)  
Vectors are orthogonal if ⟨x, y⟩ = 0.

# 🔄 Orthonormal Basis

A set of vectors is orthonormal if:  
• Each vector has unit length (‖v‖ = 1)  
• Every pair of vectors is orthogonal (⟨vᵢ, vⱼ⟩ = 0 for i ≠ j)  
Any vector in the space can be expressed uniquely as a linear combination of the orthonormal basis vectors.